Examen MMP L3 S5 du 16/12/2011

1) Nous avons

$P_0(x) = 1$

$P_1(x) = x + a$

$P_2(x) = x^2 + bx + c$

$P_3(x) = x^3 + \alpha x^2 + \beta x + \gamma$

Relations d'orthogonalité :

$$(P_0, P_1) = 0 = \int_{-1}^{1} \underbrace{(1+x^2)}_{\omega(x)} \cdot \underbrace{1}_{P_0(x)} \cdot \underbrace{(x+a)}_{P_1(x)} \, dx =$$

$$= a \int_{-1}^{1} (1+x^2)\,dx \longrightarrow \boxed{a=0} \quad [\text{donc } P_1(x) = x]$$

$$(P_0, P_2) = 0 = \int_{-1}^{1} (1+x^2)\cdot 1 \cdot (x^2+bx+c)\,dx =$$

$$= \int_{-1}^{1} (1+x^2)\,x^2\,dx + c\int_{-1}^{1}(1+x^2)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^5}{5}\right)\Big|_{-1}^{1} + c\left(x + \frac{x^3}{3}\right)\Big|_{-1}^{1} =$$

$$= \left(\frac{2}{3} + \frac{2}{5}\right) + c\left(2 + \frac{2}{3}\right) = \frac{16}{15} + \frac{8}{3}c \Rightarrow \boxed{c = -\frac{2}{5}}$$

$$(P_1, P_2) = 0 = \int_{-1}^{1} (1+x^2)\cdot x \cdot \left(x^2 + bx - \frac{2}{5}\right) =$$

$$= b\int_{-1}^{1} (1+x^2)\,x^2\,dx \Rightarrow \boxed{b = 0}$$

$$\left[\text{donc } P_2(x) = x^2 - \frac{2}{5}\right]$$

$$(P_0, P_3) = 0 = \int_{-1}^{1} (1+x^2)\cdot 1 \cdot (x^3 + \alpha x^2 + \beta x + \gamma)\,dx =$$

$$= \alpha\int_{-1}^{1} x^2(1+x^2)\,dx + \gamma\int_{-1}^{1}(1+x^2)\,dx = \frac{16}{15}\alpha + \frac{8}{3}\gamma$$

$$(P_2, P_3) = 0 = \int_{-1}^{1} (1+x^2)\cdot\left(x^2 - \frac{2}{5}\right)\cdot(x^3 + \alpha x^2 + \beta x + \gamma)\,dx =$$

$$= \int_{-1}^{1} (1+x^2)\left(x^2 - \frac{2}{5}\right)(\alpha x^2 + \gamma)\,dx =$$

$$= \alpha\int_{-1}^{1} (1+x^2)\left(x^2 - \frac{2}{5}\right)x^2\,dx + \gamma\,\underbrace{(P_0, P_2)}_{0} =$$

$$= \alpha\int_{-1}^{1} (1+x^2)\left(x^2 - \frac{2}{5}\right)x^2\,dx \longrightarrow \boxed{\alpha = \gamma = 0}$$

$$(P_1, P_3) = 0 = \int_{-1}^{1} (1+x^2)\, x\,(x^3+\beta x)\,dx = \int_{-1}^{1} x^4(1+x^2)\,dx + \beta\int_{-1}^{1} x^2(1+x^2)\,dx =$$

$$= \left(\frac{x^5}{5}+\frac{x^7}{7}\right)\Big|_{-1}^{1} + \beta\cdot\frac{16}{15} = \frac{24}{35}+\frac{16}{15}\beta \Rightarrow \beta = -\frac{9}{14}$$

$$\left[\text{donc } P_3(x) = x^3 - \frac{9}{14}x\right]$$

Dernière question :

$x^2 P_6'(x)$ est un polynôme de degré 7, donc une combinaison linéaire de $P_0(x) \dots P_7(x)$, qui sont tous orthogonaux à $P_8(x)$. D'où $(P_8(x), x P_6'(x)) = 0$.

2) $$\int_0^1 t^{3/2}(1-t^2)^{3/4}\,dt = \left| \begin{matrix} x = t^2 \\ dx = 2t\,dt \end{matrix} \right| =$$

$$= \int_0^1 x^{3/4}(1-x)^{3/4}\frac{dx}{2\sqrt{x}} = \frac{1}{2}\int_0^1 x^{1/4}(1-x)^{3/4}\,dx =$$

$$= \frac{1}{2}B\left(\frac{5}{4},\frac{7}{4}\right) = \frac{1}{2}\frac{\Gamma(5/4)\Gamma(7/4)}{\Gamma(3)} = \frac{1}{2}\frac{\frac{1}{4}\Gamma(1/4)\cdot\frac{3}{4}\Gamma(3/4)}{2!} =$$

$$= \frac{3}{64}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{3}{64}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{3\pi}{32\sqrt{2}}$$

3) $$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx = \int_{-\pi}^{\pi} e^{-i\omega x}\sin x\,dx$$

$$= \frac{1}{2i}\int_{-\pi}^{\pi} e^{-i\omega x}(e^{ix}-e^{-ix})\,dx =$$

$$= \frac{1}{2i}\int_{-\pi}^{\pi}\left(e^{-i(\omega-1)x} - e^{-i(\omega+1)x}\right)dx =$$

$$= \frac{1}{2i}\left(\frac{e^{-i(\omega-1)x}}{-i(\omega-1)} + \frac{e^{-i(\omega+1)x}}{i(\omega+1)}\right)\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{1}{2i}\left(\frac{e^{-i\omega\pi}-e^{i\omega\pi}}{i(\omega-1)} - \frac{e^{-i\omega\pi}-e^{i\omega\pi}}{i(\omega+1)}\right) = \frac{\cancel{2i}\sin\omega\pi}{\cancel{2i}}\left(\frac{1}{i(\omega+1)}-\frac{1}{i(\omega-1)}\right)$$

$$= +\frac{2i\sin\omega\pi}{\omega^2-1}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2i\sin\omega\pi}{\omega^2-1}e^{i\omega x}\,d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2i\sin\omega\pi}{\omega^2-1}(\cos\omega x + i\sin\omega x)\,d\omega =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\omega\pi\sin\omega x}{1-\omega^2}\,d\omega$$

Donc

$$\int_{-\infty}^{\infty} \frac{\sin \pi\omega \sin \omega x}{1-\omega^2}\, d\omega = \pi f(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin \pi\omega \sin \frac{\pi\omega}{6}}{1-\omega^2}\, d\omega = \pi f\left(\frac{\pi}{6}\right) = \pi \sin \frac{\pi}{6} = \frac{\pi}{2}$$

4) $\begin{cases} g'' - 8g' + 7g = \cos x \\ g(0) = g'(0) = 0 \end{cases}$

1). Fonctions de Green

$g'' - 8g' + 7g = 0$

$\Rightarrow \lambda^2 - 8\lambda + 7 = 0$

$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 7$

$\Rightarrow g_1 = e^x,\ g_2 = e^{7x}$

$W(g_1(x), g_2(x)) = e^x \cdot 7e^{7x} - e^x \cdot e^{7x} = 6e^{8x}$

Donc

$$G(x,y) = \frac{e^x \cdot e^{7y} - e^{7x} e^y}{6e^{8y}} = -\frac{1}{6}\left(e^{x-y} - e^{7x-7y}\right)$$

$$g(x) = \int_0^x G(x,y) f(y)\, dy = -\frac{1}{6} e^x \int_0^x e^{-y} \cos y\, dy + \frac{1}{6} e^{7x} \int_0^x e^{-7y} \cos y\, dy =$$

$$= -\frac{1}{6} e^x \left(\frac{e^{-y}}{2} (\sin y - \cos y) \right)\Big|_0^x + \frac{1}{6} e^{7x} \left(\frac{e^{-7y}}{50} (\sin y - 7\cos y) \right)\Big|_0^x =$$

$$= -\frac{1}{12}(\sin x - \cos x) - \frac{1}{12} e^x + \frac{1}{300}(\sin x - 7\cos x) + \frac{7}{300} e^{7x} =$$

$$= -\frac{2}{25} \sin x + \frac{3}{50} \cos x - \frac{1}{12} e^x + \frac{7}{300} e^{7x}$$

2). Transformation de Laplace

$G(s) = \int_0^{\infty} e^{-st} g(t)\, dt$

$\Rightarrow \int_0^{\infty} e^{-st} g'(t)\, dt = s G(s)$, $\int_0^{\infty} e^{-st} g''(t) = s^2 G(s)$

$(s^2 - 8s + 7) G(s) = TL(\cos t) = \frac{s}{1+s^2}$

$$\Rightarrow G(s) = \frac{s}{(s-1)(s-7)(s^2+1)} = \frac{\alpha}{s-1} + \frac{\beta}{s-7} + \frac{\gamma s + \delta}{s^2+1}$$

$$= \frac{\alpha(s^3 - 7s^2 + s - 7) + \beta(s^3 - s^2 + s - 1) + \gamma(s^3 - 8s^2 + 7s) + \delta(s^2 - 8s + 7)}{(s-1)(s-7)(s^2+1)}$$

d'où

$$\begin{cases} \alpha+\beta+\gamma=0 \\ -7\alpha-\beta-8\gamma+\delta=0 \\ \alpha+\beta+7\gamma-8\delta=1 \\ -7\alpha-\beta+7\delta=0 \end{cases} \Rightarrow \begin{cases} \alpha=-\frac{1}{12} \\ \beta=\frac{7}{300} \\ \gamma=\frac{3}{50} \\ \delta=-\frac{2}{25} \end{cases}$$

$$G(s)=-\frac{1}{12}\cdot\underbrace{\frac{1}{s-1}}_{TL(e^t)}+\frac{7}{300}\cdot\underbrace{\frac{1}{s-7}}_{TL(e^{7t})}+\frac{3}{50}\cdot\underbrace{\frac{s}{s^2+1}}_{TL(\cos t)}-\frac{2}{25}\cdot\underbrace{\frac{1}{s^2+1}}_{TL(\sin t)}$$

et on retrouve la même expression

$$g(t)=TL^{-1}(G(s))=-\frac{1}{12}e^t+\frac{7}{300}e^{7t}+\frac{3}{50}\cos t-\frac{2}{25}\sin t.$$

S) 1). $\overline{x(t)}=\sum_{n\in\mathbb{Z}}\overline{c_n}\,e^{-2\pi i nt/L}=\sum_{n\in\mathbb{Z}}\overline{c_{-n}}\,e^{2\pi i nt/L}$

$\quad \overset{\shortparallel}{x(t)}=\sum_{n\in\mathbb{Z}}c_n e^{2\pi i nt/L}$

Donc $\overline{c_n}=c_{-n}$ et, de même, $\overline{d_n}=d_{-n}$

2). $L=\int_0^L\sqrt{x'(t)^2+y'(t)^2}\,dt \underset{\text{car } x'^2+y'^2=1}{=} \int_0^L[(x')^2+(y')^2]\,dt \ \circledast$

$$\int_0^L(x')^2dt=\int_0^L\sum_{n\in\mathbb{Z}}c_n\frac{2\pi i n}{L}e^{2\pi i nt/L}\sum_{n'\in\mathbb{Z}}c_{n'}\frac{2\pi i n'}{L}e^{2\pi i n't/L}\,dt=$$

$$=\sum_{n,n'\in\mathbb{Z}}c_nc_{n'}\left(-\frac{4\pi^2nn'}{L^2}\right)\underbrace{\int_0^L e^{2\pi i(n+n')t/L}\,dt}_{\overset{\shortparallel}{L\,\delta_{n,-n'}}}=$$

$$=\sum_{n,n'\in\mathbb{Z}}c_nc_{n'}\left(-\frac{4\pi^2nn'}{L}\right)\delta_{n,-n'}=\sum_{n\in\mathbb{Z}}c_nc_{-n}\frac{4\pi^2n^2}{L}=$$

$$=|\text{grâce à 1).}|=\sum_{n\in\mathbb{Z}}|c_n|^2\frac{4\pi^2n^2}{L}$$

De même, $\int_0^L y'^2(t)\,dt=\sum_{n\in\mathbb{Z}}|d_n|^2\cdot\frac{4\pi^2n^2}{L}$

$$\circledast\ \frac{4\pi^2}{L}\sum_{n\in\mathbb{Z}}n^2(|c_n|^2+|d_n|^2).$$

3). D'après la formule de Green $\left(\int_\gamma P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \right.$

$$\int_\gamma y\,dx = -\int_D dx\,dy = -A$$

mais

$$\int_\gamma y\,dx = \int_0^L y(t)\,x'(t)\,dt = \int_0^L \sum_{n\in\mathbb{Z}} d_n e^{2\pi i n t/L} \sum_{n'\in\mathbb{Z}} c_{n'} \frac{2\pi i n'}{L} e^{2\pi i n' t/L}\,dt$$

$$= \sum_{n,n'\in\mathbb{Z}} d_n c_{n'} \frac{2\pi i n'}{L} \int_0^L e^{2\pi i (n+n')t/L}\,dt =$$

$$= \sum_{n\in\mathbb{Z}} d_n c_{-n} (-2\pi i n) = -2\pi i \sum_{n\in\mathbb{Z}} n\, c_{-n} d_n =$$

$$= -2\pi i \sum_{n\in\mathbb{Z}} n\, \bar{c}_n d_n.$$

4). $0 \le (|c_n| - |d_n|)^2 = |c_n|^2 + |d_n|^2 - 2|c_n||d_n| \Rightarrow |c_n|^2 + |d_n|^2 \ge 2|c_n||d_n|.$

Donc

$$\frac{A}{\pi} \le \sum_{n\in\mathbb{Z}} |2in\bar{c}_n d_n| \le \sum_{n\in\mathbb{Z}} (|c_n|^2 + |d_n|^2)\, n^2 =$$

$$= \frac{L^2}{4\pi^2} \Longrightarrow A \le \frac{L^2}{4\pi}$$

5). Égalité est possible $\left(A = \frac{L^2}{4\pi}\right)$ seulement si $\forall n \in \mathbb{Z}\setminus\{0\}$

$2|\bar{c}_n d_n| = |c_n|^2 + |d_n|^2$, d'où $|c_n|^2 = |d_n|^2$

et en plus $|n| = n^2 \Longrightarrow n = 0, -1, 1$

Alors

$$\begin{cases} x(t) = c_0 + c_1 e^{2\pi i t/L} + \bar{c}_1 e^{-2\pi i t/L} \\ y(t) = d_0 + c_1 e^{i\alpha} e^{2\pi i t/L} + \bar{c}_1 e^{-i\alpha} e^{-2\pi i t/L} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x(t) - c_0 = |c_1| \left(e^{2\pi i t/L + i\arg(c_1)} + e^{-2\pi i t/L - i\arg(c_1)}\right) = \\ \quad = |c_1| \cos\left(\frac{2\pi t}{L} + \arg c_1\right) \\ y(t) - d_0 = |c_1| \cos\left(\frac{2\pi t}{L} + \arg c_1 + \alpha\right) \end{cases}$$

En plus $x'^2 + y'^2 = 1$, donc

$$\begin{cases} c_1^2 + c_1^2 e^{2i\alpha} = 0 \\ \bar{c}_1^2 + \bar{c}_1^2 e^{-2i\alpha} = 0 \\ |c_1|^2\, 4\pi^2/L^2 = 1 \end{cases} \begin{matrix} \Longrightarrow \alpha = \pm\pi/2 \\ \Longrightarrow |c_1| = \frac{L}{2\pi} \end{matrix}$$

et finalement

$$\begin{cases} x(t) - c_0 = \dfrac{L}{2\pi} \cos\left(\dfrac{2\pi t}{L} + \arg c_1\right) \\ y(t) - d_0 = \dfrac{L}{2\pi} \cos\left(\dfrac{2\pi t}{L} + \arg c_1 \pm \dfrac{\pi}{2}\right) = \mp \dfrac{L}{2\pi} \sin\left(\dfrac{2\pi t}{L} + \arg c_1\right) \end{cases}$$

$$[x(t) - c_0]^2 + [y(t) - d_0]^2 = \frac{L^2}{4\pi^2}$$

l'équation du cercle de rayon $\frac{L}{2\pi}$ et de centre (c_0, d_0)